Math 246A Lecture 23 Notes

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1 Harmonic Measure

1.1 Harmonic measure

Let Ω be a bounded domain, and suppose that the Dirichlet problem is solvable in Ω. In other words, if $f \in C(\Omega)$, then there exists some continuous u_f on $\overline{\Omega}$ such that $u_f = f$ on $\partial\Omega$ and $\Delta u_f = 0$ on Ω.

Suppose we send $f \mapsto u_f(z)$ for $z \in \Omega$. This is a bounded linear functional (bounded by the maximum principle). Then there exists a probability measure μ in $\partial\Omega$ such that

$$u_f(z) = \int_{\partial\Omega} f(\zeta) d\mu_z(\zeta),$$

where

$$d\mu_z = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \, d\theta.$$

1.2 Probability measures on tori and conditional probability

Let $\mathbb{T} = \{z : |z| - 1\}$, and let $\mathbb{T}^n = \{(e^{it_1}, \dots, e^{it_n}) : e^{it_j} \in \mathbb{T}\}$ by the n-torus. This has a natural measure $d\theta_1 \cdots d\theta_n/(2\pi)^n$. We can define $\mathbb{T}^\infty = \{e^{i\theta_1}, \dots) : e^{i\theta_j} \in \mathbb{T}\}$. This has a product topology given to it from the copies of $\mathbb{T}$. In fact, it is a compact metric space. Define the projection $\pi_n(e^{i\overline{\theta}}) = (e^{i\theta_1}, \dots, e^{i\theta_n})$. The topology is the smallest topology making all the π_n continuous.

Let E be Borel, and let $E \subseteq \mathbb{T}^\infty$. There exists a product measure P_n on $\mathbb{T}^n$ for each n. Then, by the Kolmogorov extension theorem, there exists a measure P on $\mathbb{T}^\infty$ given by

$$P(E) := \lim_{n \to \infty} P_n(\pi_n(E)).$$

Let X be a measurable function on $\mathbb{T}^\infty$, and suppose that $X \in L^1$ (it is a random variable with finite expectation). Let $\mathcal{F}_n$ be the set of measurable functions of $(e^{i\theta_1}, \dots, e^{i\theta_n})$. Then the conditional expectation is

$$E[X \mid \mathcal{F}_n](e^{i\theta_1}, \dots, e^{i\theta_n}) = \int_{\mathbb{T}^\infty} X(e^{i\theta_1}, \dots, e^{i\theta_n}, e^{i\overline{\theta}}) \, dP(\overline{\theta}).$$

Note that $E[X \mid \mathcal{F}_n] \in \mathcal{F}_n$. Here is another interpretation. If $X \in L^2$, then it also acts on the Hilbert space. Then the conditional expectation is the best guess of X acting on the subspace $\mathcal{F}_n$.

1.3 Martingales in a domain

Definition 1.1. A **martingale** is a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables such that

1. $X_n \in \mathcal{F}_n$
2. $E[X_{n+1} \mid \mathcal{F}_n] = X_n$

Think of this as a fair game.

Theorem 1.1 (Doob)**.** *If $1 < p \leq \infty$, (X_n) is a martingale, and $\sup_n E[|S_n|^p] < \infty$, then there exists a measurable function X such that $X_n \to X$ a.s. and $X_n = E[X \mid \mathcal{F}_n]$ for each $n \in \mathbb{N}$.*

Returning to our bounded domain Ω, let $\overline{\theta} \in \mathbb{T}^\infty$. Let $r_0 = (1/2)\operatorname{dist}(z_0, \partial\Omega)$. Define $z_1 = z_0 + r_0 e^{i\theta_1}$. Now let $r_1 = \frac{1}{2}\operatorname{dist}(z_1, \partial\Omega)$, and let$z_2 = z_1 + r_1 e^{i\theta_2}$. Continuing this, we can get a sequence of z_n and r_n such that z_n and r_n are $\mathcal{F}_n$ measurable and $E[z_{n+1} \mid \mathcal{F}_n] = z_n$. So the (z_n) form a bounded martingale.

By Doob's martingale theorem, $z_n \xrightarrow{a.s.} z_\infty$ for some z_∞. This means that $r_n = |z_{n+1} - z_n| \to 0$ a.s., as well. So $z_\infty \in \partial\Omega$ a.s. Now let $f \in C(\partial\Omega)$, and let $u_f(z_n) = Z_n$. Now let $X_{n+1} = u_f(z_n + r_n e^{i\theta_n})$, and observe that

$$E[X_{n+1} \mid \mathcal{F}_n] = \frac{1}{2\pi}\int u_f(z_n + r_n e^{i\theta_{n+1}})\, d\theta_{n+1} = u_f(z_n).$$

By continuity, $X_n \to f(z_\infty)$. This means that

$$\int f(\zeta)\, dP = u_f(z_0),$$

so $\mu_{z_0}(E) = P(z_\infty \in E)$.

Corollary 1.1. *Let $v \in C(\overline{\Omega})$ be such that for all $z \in \Omega$,*

$$v(z) = \frac{1}{2\pi}\int_0^{2\pi} v(z + r_z e^{i\theta})d\,\theta$$

where $r_z = \frac{1}{2}\operatorname{dist}(z, \partial\Omega)$. Then v is harmonic.

Remark 1.1. We don't actually need $r_z = \frac{1}{2}\operatorname{dist}(z, \partial\Omega)$. We can pick r_z such that $|z_{n+1} - z_n| \to 0$, such as $r(z) = \frac{9}{10}\operatorname{dist}(z, \partial\Omega)$.

$z_{n+1} = z_n + \frac{1}{2}\,\mathrm{dist}(z_n, \partial\Omega)e^{i\theta_{n+1}}$. Observe that

$$\int \underbrace{(z_{n+1} - z_n)}_{re^{i\theta_{n+1}}}\underbrace{\overline{(z_{m+1} - z_m)}}_{r_m e^{-i\theta_{m+1}}}\, dP = \delta_{n,m} r_n^2.$$

This tells us that

$$\mathbb{E}\left[\sum |z_{n+1} - z_n|^2\right] = r_n^2.$$